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# Beyond the singularity of the $2D$ charged black hole

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Abstract: Two dimensional charged black holes in string theory can be obtained as exact $\frac{\mathrm{SL}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)}$ quotient CFTs. The geometry of the quotient is induced from that of the group, and in particular includes regions beyond the black hole singularities. Moreover, wavefunctions in such black holes are obtained from gauge invariant vertex operators in the $\mathrm{SL}(2,\mathbb{R})$ CFT, hence their behavior beyond the singularity is determined. When the black hole is charged we find that the wavefunctions are smooth at the singularities, and known results of infinite blue shifts at the horizons persist. Unlike the uncharged case, scattering waves prepared beyond the singularity are not fully reflected; part of the wave is transmitted through the singularity. Hence, the physics outside the horizon of a *charged* black hole is sensitive to conditions set behind the past singularity.

Keywords: Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence, Black Holes in String Theory.

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## 1. Introduction

The two dimensional black hole in string theory [1]–[3] is obtained as an exact $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ quotient CFT background [2]. The geometry of this 2-$d$ black hole is similar to a two dimensional slice of the Schwarzschild solution, whose maximal analytic extension is shown in figure 1$a$ in a Kruskal diagram and its Penrose diagram is shown in figure 1$b$. In the 4-$d$ case every point in figure 1 is actually a two sphere, while in the 2-$d$ case there is a non-trivial dilaton.

In general relativity, regions beyond the singularities are not considered usually, since initial data set outside the horizon of the black hole cannot determine the behavior of classical solutions beyond the singularities. In particular, wavefunctions are singular at the singularity of the Schwarzschild(-like) black hole. On the other hand, the geometry of the $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ quotient CFT background is induced from the geometry of $\mathrm{SL}(2,\mathbb{R})$, and as a consequence includes regions beyond the singularities. Moreover, T-duality interchanges the region outside the horizon of the black hole with the one beyond the singularity [4, 3] (for a review, see [5]).

In string theory one is thus led to investigate the inclusion of the regions beyond the singularity. A natural question to ask is: What happens if one prepares a scattering wave beyond the singularity? The answer to this question is given unambiguously by doing the following. The wavefunctions in the quotient background are obtained from gauge invariant vertex operators in the $\mathrm{SL}(2,\mathbb{R})$ CFT. As we shall describe later, one can obtain in this way the behavior of scattering waves. In the uncharged 2-$d$ case it was found [3] that low energy scattering waves, obtained this way beyond the singularity, are fully reflected from the singularity. We interpret this result as a confirmation that the regions beyond the singularities of the uncharged black hole can be ignored (as far as a low energy physics outside the black hole is concerned).

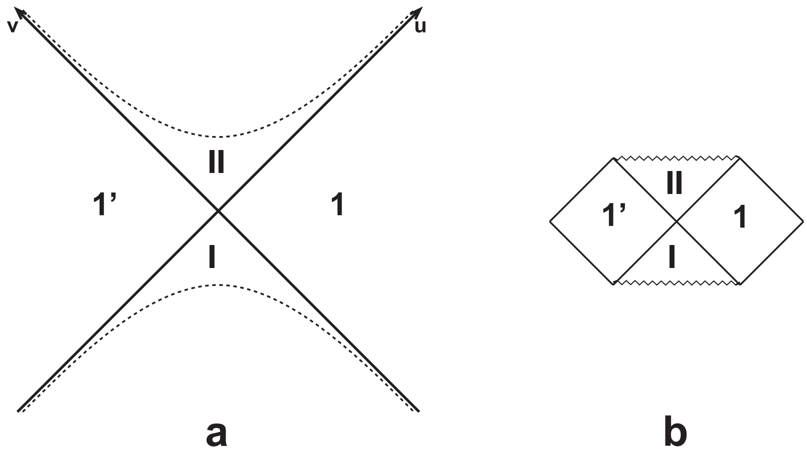


**Figure 1:** *a* Kruskal diagram and *b* Penrose diagram of the Schwarzshild or the 2-*d* black hole.



In this note we consider the same issues for the two dimensional *charged* black hole of [6]; related solutions appear in [7]–[12]. It is very reminiscent of the maximally extended Reissner-Nordstrom solution whose Penrose diagram is shown in figure 2. In string theory such backgrounds are obtained from a family of exact $\frac{\widehat{\mathrm{SL}}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)}$ quotient CFT sigma models, where $\widehat{\mathrm{SL}}(2,\mathbb{R})$ is the universal cover of $\mathrm{SL}(2,\mathbb{R})$. Again, in the 4-*d* case each point in figure 2 is actually a two sphere, while in the 2-*d* case there is a non-trivial dilaton. The singularity is time-like and may be avoided by free falling neutral probes (see for instance [8, section 3]). We investigate if the singularity has other mild aspects in classical string theory.

The geometry of the quotient CFT background is induced again from the geometry of $\widehat{\mathrm{SL}}(2,\mathbb{R})$ and, therefore, in string theory we should consider including the regions beyond the singularities. Moreover, scattering waves are uniquely determined from vertex operators in the $\mathrm{SL}(2,\mathbb{R})$ CFT. By following this route we will show that scattering waves prepared beyond the singularity are *not* fully reflected if the black hole is charged. Part of the wavefunction is transmitted to regions outside the black hole, in a manner consistent with unitarity. As a consequence, initial data prepared beyond the singularities affects observers outside the horizon of a charged black hole.

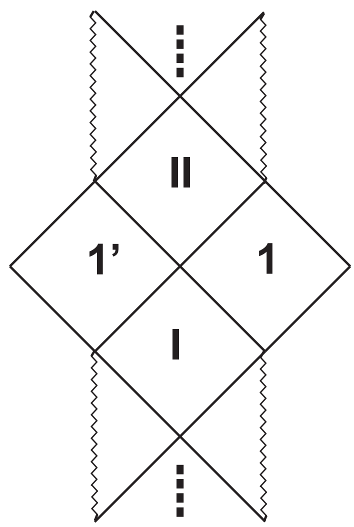


**Figure 2:** Penrose diagram of the Reissner-Nordstrom black hole.

In section 2, we present a family of three dimensional backgrounds obtained by gauging the WZW model of the four dimensional $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ group manifold by a family of non-compact $\mathrm{U}(1)$ subgroups. Then, by taking a small constant radius of the $\mathrm{U}(1)$ part, we obtain the two dimensional charged black hole via the Kaluza-Klein (KK) mechanism. All two dimensional spaces have asymptotically flat regions before event horizons

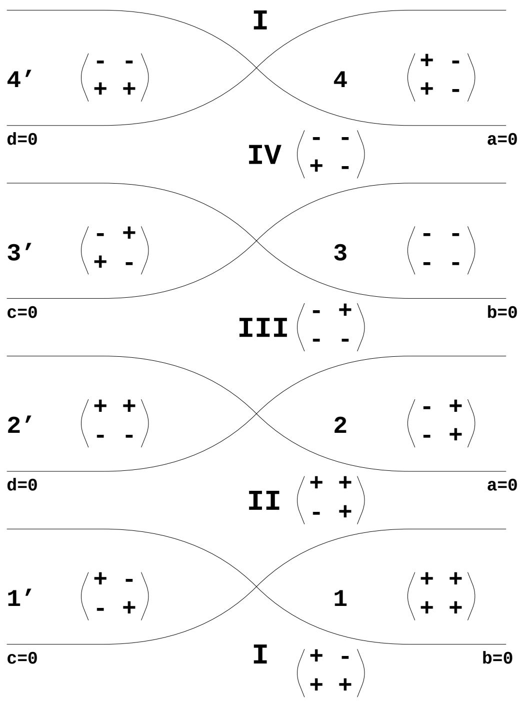


**Figure 3:** A two dimensional slice of $\mathrm{SL}(2,\mathbb{R})$.

and beyond inner horizons, as well as regions between the event horizons and the inner horizons. They admit singularities which are generated by null gauge identifications and lie behind the inner horizons. These backgrounds also have a non-trivial dilaton and a background KK gauge field, and they can be obtained by $O(1,2) \subset O(2,2)$ rotations along the lines of [10] (for a review, see [5]).

In section 3, we discuss vertex operators in the quotient CFT. In section 4, we present a class of solutions to the wave equation in the $\frac{\mathrm{SL}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)}$ CFT geometry which describe scattering waves from beyond a singularity. One finds that such scattering waves correspond to the vertex operators in the quotient CFT. Moreover, we find that if the black hole is charged, part of the wave which is coming from beyond the singularity leaks into the region outside the horizon of the black hole. From the point of view of the wavefunctions there is nothing special at the location of the singularity. The results are discussed in section 5. In the appendix, we present the behavior of the wave scattered from beyond the singularity in all the various regions of the maximally extended charged black hole.

## 2. The geometry of $\frac{\mathrm{SL}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)}$

In this section we shall describe the geometry of the coset $\frac{\mathrm{SL}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)}$ and its universal cover $\frac{\widehat{\mathrm{SL}}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)}$. We construct a three dimensional background by gauging [13] the WZW model of the four dimensional $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ group manifold by a non-compact $\mathrm{U}(1)$ subgroup. Let $(g, x) \in \mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ be a point on the product group manifold where $x \sim x + 2\pi L$, and let $k > 0$ be the level of $\mathrm{SL}(2,\mathbb{R})$ (such that its signature is

$(++-))$. The U(1) gauge group acts as

$$(g, x_L, x_R) \to \left(e^{\rho\sigma_3/\sqrt{k}} g e^{\tau\sigma_3/\sqrt{k}}, x_L + \rho', x_R + \tau'\right), \tag{2.1}$$

where $x_{L,R}$ are the left-handed and right-handed parts of $x$, respectively. Since we gauge only a single U(1) out of the two right-handed U(1) generators in (2.1), the two parameters $(\tau, \tau') \equiv \underline{\tau}$ are not independent but rather are constrained by

$$\underline{\tau} \equiv \tau \underline{u}, \tag{2.2}$$

where $\underline{u}$ is some unit real 2-vector. The left-handed parameters $(\rho, \rho') \equiv \underline{\rho}$ in (2.1) depend linearly on the right-handed $\underline{\tau}$ parameters. For an anomaly free gauging this dependence has to take the form

$$\underline{\rho} = R\underline{\tau}, \tag{2.3}$$

where the matrix $R$ is an $SO(2)$ matrix

$$R = \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix}. \tag{2.4}$$

The gauged action, as in [14], is then defined by

$$S = S\left[e^{\hat{\rho}\sigma_3/\sqrt{k}} g e^{\hat{\tau}\sigma_3/\sqrt{k}}\right] + S'\left[x + \hat{\rho}' + \hat{\tau}'\right] - \frac{1}{2\pi}\int d^2z \left(\partial\underline{\hat{\rho}} - R\partial\underline{\hat{\tau}}\right)^T \left(\bar{\partial}\underline{\hat{\rho}} - R\bar{\partial}\underline{\hat{\tau}}\right). \tag{2.5}$$

Here, $S[g]$ is the WZW action,

$$S[g] = \frac{k}{4\pi}\left[\int_\Sigma \mathrm{Tr}\left(g^{-1}\partial g g^{-1}\bar{\partial} g\right) - \frac{1}{3}\int_B \mathrm{Tr}\left(g^{-1}dg\right)^3\right], \tag{2.6}$$

where $\Sigma$ is the string's worldsheet and $B$ a 3-submanifold of the group $\mathrm{SL}(2,\mathbb{R})$ bounded by the image of $\Sigma$. $S'[x]$ is

$$S'[x] = \frac{1}{2\pi}\int_\Sigma \partial x \bar{\partial} x\ . \tag{2.7}$$

$\underline{\hat{\rho}}$ and $\underline{\hat{\tau}}$ are independent fields subject to the constraints

$$\underline{\hat{\tau}} = \left(\underline{\hat{\tau}}^T \underline{u}\right)\underline{u}, \qquad \underline{\hat{\rho}} = \left(\underline{\hat{\rho}}^T R\underline{u}\right) R\underline{u}. \tag{2.8}$$

The action (2.5) is invariant under the gauge transformation (2.1) for the fields $g$ and $x$ together with the field transformation

$$\begin{aligned} \underline{\hat{\rho}} &\to \underline{\hat{\rho}} - \underline{\rho} \\ \underline{\hat{\tau}} &\to \underline{\hat{\tau}} - \underline{\tau} \end{aligned} \tag{2.9}$$

provided that the parameters $\underline{\rho}$ and $\underline{\tau}$ satisfy the relation (2.3). Using the Polyakov-Wiegmann identity one sees that the action (2.5) depends on $\underline{\hat{\rho}}$ and $\underline{\hat{\tau}}$ only through the quantities

$$\begin{aligned} A &= \underline{u}^T \partial\underline{\hat{\tau}} \\ \bar{A} &= (R\underline{u})^T \bar{\partial}\underline{\hat{\rho}}. \end{aligned} \tag{2.10}$$

The gauged action has then the form

$$S = S[g] + S'[x] + \frac{1}{2\pi}\int d^2z[A\bar{\mathbf{J}}^T\underline{u} + \bar{A}\mathbf{J}^T R\underline{u} + 2A\bar{A}\,(R\underline{u})^T\, M\underline{u}]\,. \tag{2.11}$$

$A$ and $\bar{A}$ are holomorphic and anti-holomorphic gauge fields. $\mathbf{J}^T$ and $\bar{\mathbf{J}}^T$ are the row vector of currents,

$$\begin{aligned}\mathbf{J}^T &= (\sqrt{k}Tr[\partial g g^{-1}\sigma_3], 2\partial x)\\ \bar{\mathbf{J}}^T &= (\sqrt{k}\,\mathrm{Tr}[g^{-1}\bar{\partial} g\sigma_3], 2\bar{\partial} x)\,.\end{aligned} \tag{2.12}$$

The $2\times 2$ matrix $M$ in (2.11) is:

$$M = \begin{pmatrix} \frac{1}{2}\,\mathrm{Tr}[g^{-1}\sigma_3 g\sigma_3] & 0\\ 0 & 1\end{pmatrix} + R\,. \tag{2.13}$$

One can write the same action as a complete square

$$\begin{aligned} S \,=\, S[g] + S'[x] + &+\frac{1}{2\pi}\int d^2z\left[\left(\bar{A} + \frac{\bar{\mathbf{J}}^T\underline{u}}{2\,(R\underline{u})^T\, M\underline{u}}\right) 2\,(R\underline{u})^T\, M\underline{u}\left(A + \frac{\mathbf{J}^T R\underline{u}}{2\,(R\underline{u})^T\, M\underline{u}}\right) -\right.\\ &\left.-\frac{\left(\bar{\mathbf{J}}^T\underline{u}\right)\left(\mathbf{J}^T R\underline{u}\right)}{2\,(R\underline{u})^T\, M\underline{u}}\right]. \end{aligned} \tag{2.14}$$

After integrating out the fields $A$ and $\bar{A}$, to first order in $1/k$ the resulting action is:[1]

$$S = S[g] + S'[x] - \frac{1}{4\pi}\int d^2z\left[\frac{\left(\bar{\mathbf{J}}^T\underline{u}\right)\left(\mathbf{J}^T R\underline{u}\right)}{(R\underline{u})^T\, M\underline{u}}\right] \tag{2.15}$$

and the dilaton, which is normalized such that $g_s = e^{\Phi}$, becomes

$$\Phi = \Phi_0 - \frac{1}{2}\log\left((R\underline{u})^T\, M\underline{u}\right)\,. \tag{2.16}$$

Now we have to choose a specific parameterization of $\mathrm{SL}(2,\mathbb{R})$. Since the gauge group acts on a group element $g\in\mathrm{SL}(2,\mathbb{R})$ by multiplication from the left and from the right in powers of $e^{\sigma_3}$, it is convenient to represent the group elements $g\in\mathrm{SL}(2,\mathbb{R})$ by

$$g(\alpha,\beta,\theta;\epsilon_1,\epsilon_2,\delta) = e^{\alpha\sigma_3}(-1)^{\epsilon_1}(i\sigma_2)^{\epsilon_2} g_\delta(\theta) e^{\beta\sigma_3} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}, \tag{2.17}$$

where

$$\epsilon_{1,2} = 0,1\,;\qquad \delta = I,1,1'\,;\qquad ad-bc = 1\,;\qquad a,b,c,d\in\mathbb{R}\,, \tag{2.18}$$

$$g_I = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix};\qquad 0\le\theta\le\frac{\pi}{2}\,, \tag{2.19}$$

$$g_1 = g_{1'}^{-1} = \begin{pmatrix}\cosh\theta & \sinh\theta\\ \sinh\theta & \cosh\theta\end{pmatrix};\qquad 0\le\theta<\infty\,, \tag{2.20}$$

[1] In the superconformal extension this background is claimed to be exact [15].

and $\sigma_i$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.21}$$

This representation splits $\mathrm{SL}(2,\mathbb{R})$ into twelve regions[2] (see figure 3). Regions 1,1' and I are represented by $\epsilon_1 = \epsilon_2 = 0$ and $\delta = 1, 1', I$. The other nine region are represented by different $\epsilon_{1,2}$. In different regions the signs of the elements $a, b, c, d$ in $g$ (2.17) are different, as indicated in figure 3.

Define

$$W = \mathrm{Tr}(\sigma_3 g \sigma_3 g^{-1}) = 2(2ad - 1) = 2(2bc + 1)\,, \tag{2.22}$$

which is invariant under the gauge group action. Regions I,II,III and IV (type B) have $|W| \leq 2$. Regions 1,1',3 and 3' (type A) have $W > 2$. Regions 2,2',4 and 4' (type C) have $W < -2$ (for more details see [16]).

The gauge invariance of the action is fixed (for $\psi \neq \pi$) by setting

$$\alpha = -\beta \equiv \frac{1}{2} y\,. \tag{2.23}$$

After plugging (2.17) in the gauge (2.23) into (2.15), (2.16) one gets[3]

$$\begin{aligned} S &= \frac{1}{2\pi}\int_\Sigma \partial x \bar{\partial} x + \frac{k}{2\pi}\int d^2 z \left[ -\partial\theta_B \bar{\partial}\theta_B + \sin^2(\theta_B)\partial y \bar{\partial} y \right] + \\ &\quad + \frac{1}{\pi}\int d^2 z \frac{\left(\sqrt{k}\sin^2(\theta_B)\underline{u}_1 \bar{\partial} y - \underline{u}_2 \bar{\partial} x\right)\left(\sqrt{k}\sin^2(\theta_B)(R\underline{u})_1 \partial y + (R\underline{u})_2 \partial x\right)}{(R\underline{u})^T M \underline{u}} \end{aligned} \tag{2.24}$$

$$\Phi = \Phi_0 - \frac{1}{2}\log\left( (R\underline{u})^T M \underline{u} \right), \tag{2.25}$$

where $|W| \leq 2$. In the regions where $W > 2$, $\theta_B$ in (2.24), (2.25) should be replaced by $i\theta_A$. In the regions with $W < -2$, substitute $i\theta_C$ for $\theta_B - \frac{\pi}{2}$:

$$\begin{array}{cccc} B & |W| \leq 2\,, & I, II, III, IV\ : & \theta_B \\ A & W > 2\,, & 1, 1', 3, 3'\ : & \theta_B \to i\theta_A \\ C & W < -2\,, & 2, 2', 4, 4'\ : & \theta_B \to i\theta_C + \frac{\pi}{2}\,. \end{array} \tag{2.26}$$

If we take the vector

$$\underline{u}^T = (1, 0)\,, \tag{2.27}$$

then $G_{xx}$ is constant[4] and after rescaling $x \to \sqrt{k} x$ the action and the dilaton become:

[2]On the boundaries between the regions one has to use a different representation, see [16, 17].

[3]Equation (2.23) does not fix the gauge at $g_\delta = 1$ in (2.17), but it turns out that the space is well described by the coordinates (2.17) (this is not the case for the coordinates $u$ and $v$ that will be defined shortly).

[4]Actually, $G_{xx} = \mathrm{const} \iff (G + B)_{yx} = 0$ and, therefore, in this case the $\frac{\mathrm{SL}(2)\times \mathrm{U}(1)}{\mathrm{U}(1)}$ background can be used in the heterotic string.

$$
\begin{aligned}
S &= \frac{k}{2\pi}\int_\Sigma \partial x\bar\partial x + \frac{k}{2\pi}\int d^2z\left[-\partial\theta_B\bar\partial\theta_B + \sin^2(\theta_B)\partial y\bar\partial y\right] + \\
&\quad + \frac{k}{\pi}\int d^2z \frac{\sin^2(\theta_B)\bar\partial y\left(\sin^2(\theta_B)\cos(\psi)\partial y - \sin(\psi)\partial x\right)}{1+\cos(\psi)\cos(2\theta_B)} \\
&= \frac{k}{2\pi}\int d^2z\left[-\partial\theta_B\bar\partial\theta_B + \frac{\partial y\bar\partial y - 2p\bar\partial y\partial x}{\cot^2(\theta_B)+p^2} + \partial x\bar\partial x\right] \qquad (2.28)\\
\Phi &= \tilde\Phi_0 - \frac{1}{2}\log\left(\cos^2(\theta_B) + p^2\sin^2(\theta_B)\right), \qquad \tilde\Phi_0 \equiv \Phi_0 + \frac{1}{2}\log\left(\frac{1+p^2}{2}\right), \qquad (2.29)
\end{aligned}
$$

where

$$
p \equiv \tan\left(\frac{\psi}{2}\right). \qquad (2.30)
$$

Again, in regions A,C make the appropriate replacement for $\theta_B$ (see eq. (2.26)). For large $k$ and for small radius of the circle parameterized by $x$, the action (2.28) describes a two dimensional space-time parameterized by $(\theta_i, y)$, $i = A, B, C$. The three dimensional metric and antisymmetric background read from (2.24) can be reduced to two dimensions via the Kaluza-Klein mechanism. The term proportional to $\partial x\bar\partial y$ gives rise in two dimensions to a U(1) gauge field whose charge is the momentum as well as the winding along the $x$ circle. The two dimensional metric and background gauge field take the form

$$
\begin{aligned}
\frac{1}{k}ds^2 &= -d\theta_B^2 + \frac{\cot^2(\theta_B)}{\left(\cot^2(\theta_B)+p^2\right)^2}dy^2 \\
A_y &= \frac{\sqrt{k}p}{\cot^2(\theta_B)+p^2} \qquad (2.31)
\end{aligned}
$$

In regions A,C make the appropriate replacement for $\theta_B$ (see eq. (2.26)). This is the two dimensional charged black hole [6, 10]. To obtain the usual metric of the charged black hole (in Schwarzschild-like coordinates) do the following coordinate transformation: Rescale $y \to \left(1-p^2\right)t$, and define the coordinate $r$ to be a linear function of the dilaton (in regions of type A), given by

$$
r = \frac{1}{2}\log\left(\sinh^2(\theta_A) + \frac{1}{1-p^2}\right). \qquad (2.32)
$$

The metric (2.31) becomes

$$
\frac{1}{k}ds^2 = f(r)^{-1}dr^2 - f(r)dt^2 , \qquad (2.33)
$$

where

$$
\begin{aligned}
f(r) &= 1 - 2me^{-2r} + q^2e^{-4r} \\
2m &= \frac{1+p^2}{1-p^2}, \qquad q = \frac{p}{1-p^2}. \qquad (2.34)
\end{aligned}
$$

The background gauge field and dilaton are given by:

$$
A_t(r) = \sqrt{k}\left(qe^{-2r} - p\right), \qquad \Phi(r) = \tilde\Phi_0 - \frac{1}{4}\log(1-p^2)^2 - r \equiv \phi_0 - r. \qquad (2.35)
$$

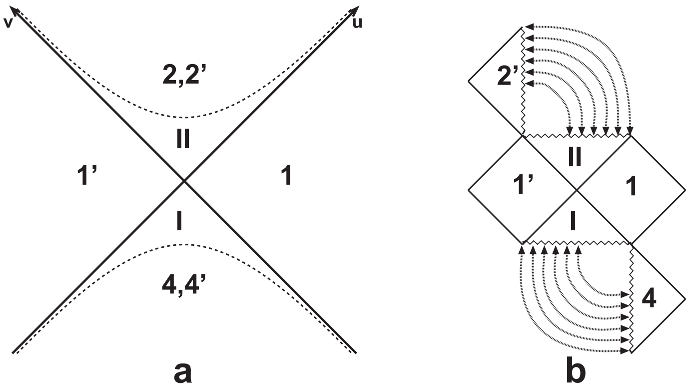


**Figure 4:** *a* The 2-dimensional black hole ($\psi = 0$); the solid lines are horizons and the dashed lines are curvature singularities. *b* The Penrose diagram of the 2-dimensional black hole. The singularity is space-like when viewed from regions I,II, and time-like from 2',4.

This metric describes a two dimensional charged black hole with mass $M$ and charge $Q$ [6]:

$$M = me^{-2\phi_0}\,, \qquad Q = qe^{-2\phi_0}\,. \tag{2.36}$$

In the "Kruskal" coordinates[5] (for $|W| \leq 2$)

$$u = \sin(\theta_B)e^{y}\,, \qquad v = \sin(\theta_B)e^{-y} \tag{2.37}$$

the metric, dilaton and gauge field (2.29)–(2.31) are:[6]

$$\begin{aligned} \frac{1}{k}ds^2 &= \frac{v^2du^2 + u^2dv^2}{4uv}\left(\frac{1-uv}{(1-(1-p^2)uv)^2} - \frac{1}{1-uv}\right) - \\ &\quad - \frac{dudv}{2}\left(\frac{1-uv}{(1-(1-p^2)uv)^2} + \frac{1}{1-uv}\right) \end{aligned} \tag{2.38}$$

$$\Phi = \tilde{\Phi}_0 - \frac{1}{4}\log\left(1-(1-p^2)uv\right)^2 \tag{2.39}$$

$$A_u = \frac{\sqrt{k}}{2}\frac{pv}{1-(1-p)\,uv}\,, \qquad A_v = -\frac{\sqrt{k}}{2}\frac{pu}{1-(1-p)\,uv}\,. \tag{2.40}$$

For the degenerate case $\psi = 0$ ($p = 0$), the metric, dilaton and background gauge field are:

$$\begin{aligned} ds^2 &= -k\frac{dudv}{1-uv} \\ \Phi &= \tilde{\Phi}_0 - \frac{1}{4}\log(1-uv)^2 \end{aligned} \qquad A_u = A_v = 0\,. \tag{2.41}$$

[5]The coordinates $u$ and $v$ cover all of PSL(2)/U(1), and only half of SL(2)/U(1) (or the Poincaré patch of the universal cover).

[6]For $\psi \neq 0$, $(\theta_B, y)$ are not good coordinates at the points $\theta_B = \pm\pi/2$ (like the origin of $\mathbb{R}^2$ in polar coordinates). As a result, the parametrization of these points is degenerated. The coordinate transformation (2.37) is singular at $\theta_B = \pm\pi/2$ and, as a result, what looks like light-like lines at $uv = 1$ are actually the points $\theta_B = \pm\pi/2$.

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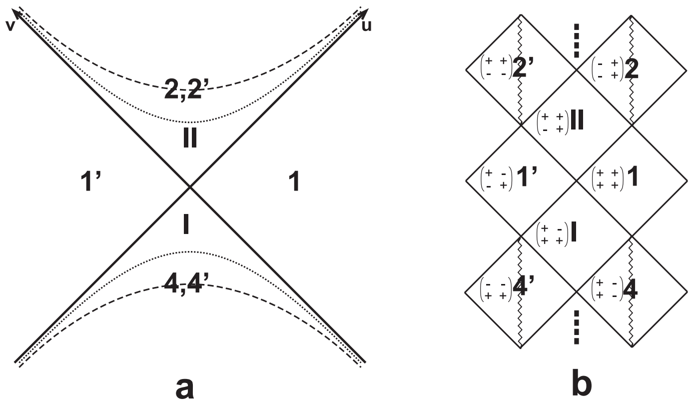


**Figure 5:** *a* The 2-dimensional charged black hole ($\psi \neq 0$); the dashed lines are curvature singularities and the doted lines (which are actually points) are horizons. *b* The Penrose diagram of the 2-dimensional charged black hole.

This is the two dimensional lorentzian black hole background. This space is plotted in figure 4. The past and future horizons are located at the $uv = 0$ lines while the singularities are located at the $uv = 1$ lines. Regions of type A,C are static and approach flat space at infinity. These are the regions before (after) the future (past) horizons at $uv < 0$ and beyond the singularities at $uv > 1$. Regions of type B are stretched between the horizons and the singularities at $0 \leq uv \leq 1$.

When we charge the black hole ($\psi, p \neq 0$), the curvature singularities move into regions of type C ($uv = \frac{1}{1-p^2} > 1$), as indicated in figure 5. The lines $uv = 1$ turn into inner horizons. For $\psi = 0$ the source of the singularity is a fixed line of the gauge transformation and for $\psi \neq 0$ the source is a null gauge orbit (see [14, appendix]). Finally, in the universal cover the geometry repeats itself.

## 3. Vertex operators in $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)}{\mathrm{U}(1)}$

Next, we would like to investigate the behavior of wavefunctions in the 2-$d$ charged black hole geometry. However, we do not allow arbitrary continuations of independent solutions across the singularities. Instead, we are guided by the structure of the vertex operators in the $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)}{\mathrm{U}(1)}$ CFT. Hence, in this section, we first discuss the (low lying) vertex operators in $\frac{\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)}{\mathrm{U}(1)}$. Vertex operators in a coset CFT are obtained by imposing the gauge conditions on those in the underlying WZW model. Thus, we begin by inspecting the low lying vertex operators in the $\mathrm{SL}(2,\mathbb{R})$ CFT, or its universal cover $\widehat{\mathrm{SL}}(2,\mathbb{R}) \equiv AdS_3$.

We are interested in the lorentzian case, but since with a euclidean worldsheet the euclidean $AdS_3 (\equiv H_3^+)$ is better behaved, we first consider euclidean $AdS_3$. The elements of $H_3^+$ can be parametrized by the $2 \times 2$ matrices [18]:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{\phi}\gamma & -e^{-\phi} - e^{\phi}\gamma\bar{\gamma} \\ e^{\phi} & -e^{\phi}\bar{\gamma} \end{pmatrix} = \begin{pmatrix} e^{-\phi} + e^{\phi}\gamma\bar{\gamma} & e^{\phi}\gamma \\ e^{\phi}\bar{\gamma} & e^{\phi} \end{pmatrix} (-i\sigma_2) \,, \qquad (3.1)$$

where $\phi \in \mathbb{R}$, $\gamma \in \mathbb{C}$, and $\bar\gamma = (\gamma)^*$. Low lying primary vertex operators are eigenfunctions of the laplacian on $H_3^+$ with eigenvalues $-h(h-1)$. They are given by [18]:

$$\Phi_h(x,\bar x;\phi,\gamma,\bar\gamma) = \frac{2h-1}{\pi}\left(|\gamma - x|^2 e^{\phi} + e^{-\phi}\right)^{-2h} . \tag{3.2}$$

The complex conjugate parameters $x, \bar x$ are interpreted as the coordinates of the euclidean $CFT_2$ dual to $AdS_3$.

Alternatively, we can write the vertex operators in "momentum space." We are interested in eigenfunctions of the transformation $g \to e^{\alpha\sigma_3} g e^{\beta\sigma_3}$ with eigenvalues $e^{2i(m\alpha+\bar m\beta)}$, $m, \bar m \in \mathbb{R}$, and those are given by:[7]

$$K_{j;m,\bar m}(g) = \int d^2x x^{j+im}\bar x^{j-i\bar m}\Phi_{j+1}(x,\bar x;\phi,\gamma,\bar\gamma) . \tag{3.3}$$

For future purposes we shall inspect the asymptotic behavior of $K_{j;m,\bar m}$ at large $\phi$ and for $j+\frac{1}{2} \in i\mathbb{R}$. Using the asymptotic behavior of $\Phi_h(x,\bar x;\phi,\gamma,\bar\gamma)$ [18] and eq. (3.3), one finds:

$$\begin{aligned} K_{j;m,\bar m}(g(\phi\to\infty)) &\sim \gamma^{im+j}\bar\gamma^{-i\bar m+j}e^{2j\phi} + R(j;m,\bar m)\gamma^{im-(j+1)}\bar\gamma^{-i\bar m-(j+1)}e^{-2(j+1)\phi} \\ &\sim \left(\frac{-a}{d}\right)^{\frac{i}{2}(m+\bar m)}\left(\frac{-b}{c}\right)^{\frac{i}{2}(m-\bar m)}\left[(-ad)^j + R(j;m,\bar m)(-ad)^{-(j+1)}\right] , \end{aligned} \tag{3.4}$$

with

$$R(j;m,\bar m) = \frac{\Gamma(j+1+im)\Gamma(j+1+i\bar m)\Gamma(-2j-1)}{\Gamma(-j+im)\Gamma(-j+i\bar m)\Gamma(2j+1)} . \tag{3.5}$$

We now return to lorentzian signature. The analytic continuation from euclidean to lorentzian $AdS_3$ is obtained by taking $\gamma$ and $\bar\gamma$ to be independent real parameters. In this case $a, b, c, d$ in eq. (3.1)are real and cover a Poincaré patch of $AdS_3$. In the parametrization of eq. (2.17), large $\phi$ corresponds to large $\theta$. For large $\theta$, near the boundary of $\mathrm{SL}(2,\mathbb{R})$, eq. (2.17)reads:

$$g(\theta\to\infty) \sim e^{\theta}\begin{pmatrix} \pm e^{\alpha+\beta} & \pm e^{\alpha-\beta} \\ \pm e^{\beta-\alpha} & \pm e^{-\alpha-\beta}\end{pmatrix} . \tag{3.6}$$

Hence, in the gauge (2.23), and for $j = -\frac{1}{2} + is$, the asymptotic behavior of $K_{j;m,\bar m}$ is:

$$K_{j;m,\bar m}\left(g(\theta\to\infty)\right) \sim e^{-\theta+iy(m-\bar m)}\left[e^{i2s\theta} + R(j;m,\bar m)e^{-i2s\theta}\right] . \tag{3.7}$$

After adding the U(1) contribution, the vertex operator in $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$ becomes:

$$V^j_{m,k_L;\bar m,k_R} = K_{j;m,\bar m}(g)e^{i(k_L x_L + k_R x_R)} . \tag{3.8}$$

Applying the gauge transformation (2.1) to $(g, x)$, the operator $V^j_{m,k_L;\bar m,k_R}$ gets multiplied by the phase

$$e^{i\left(\frac{m}{\sqrt{k}}\rho + k_L\rho' + \frac{\bar m}{\sqrt{k}}\tau + k_R\tau'\right)} . \tag{3.9}$$

[7] These are the same as the operators usually considered in CFT on $AdS_3$, but with $m$ replaced by $im$ and $\bar m$ by $-i\bar m$ (see the discussion in [16, section 2.2]).

In the $\frac{\mathrm{SL}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)}$ coset, only those vertex operators for which this phase equals to 1 are allowed. Taking (2.3),(2.2) into account we get a constraint on the allowed charges $(m, k_L; \bar{m}, k_R)$; in matrix notation it reads:

$$\left[\left(\frac{m}{\sqrt{k}}, k_L\right) R + \left(\frac{\bar{m}}{\sqrt{k}}, k_R\right)\right] \underline{u} = 0\,, \tag{3.10}$$

where $(k_L, k_R) \in \Gamma^{1,1}$ are quantized on the even self-dual Narain lattice. For the choice (2.27), the constraint reads:

$$\bar{m} + m\cos(\psi) - \sqrt{k}k_L \sin(\psi) = 0\,. \tag{3.11}$$

Finally, in string theory on $\frac{\mathrm{SL}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)} \times \mathcal{N}$, physical vertex operators are given by the dressing of operators in the internal CFT $\mathcal{N}$ with $V^{j}_{m,k_L;\bar{m},k_R}$, such that the on-shell string conditions are obtained. The precise details of the on-shell conditions depend on the type of string theory considered.

## 4. Scattering from behind the singularity

In this section, we will show that vertex operators in the $\frac{\mathrm{SL}(2,\mathbb{R})\times \mathrm{U}(1)}{\mathrm{U}(1)}$ coset CFT describe scattering waves in the 2-$d$ charged black hole geometry. Our focus will be on waves which are incoming from a region behind a singularity, say region 4 in figure 5 (though the discussion is easily extended to scattering waves from asymptotically flat regions outside the black hole, like region 1). For this purpose, we first discuss independent wave solutions in $\mathrm{SL}(2,\mathbb{R})$, and then consider particular linear combinations which, when reduced to the coset, describe scattering waves incoming entirely from region 4. These scattering waves will turn out to correspond to the vertex operators in the coset CFT discussed in section 3.

A general wave in $L_2\left(\widehat{\mathrm{SL}}(2,\mathbb{R})\right)$ is given by a linear combination of $\widehat{\mathrm{SL}}(2,\mathbb{R})$ matrix elements in the principal continuous and discrete series representations. Matrix elements of $g$ in a representation with a Casimir $-j(j+1)$, $\mathbf{K}(j;g)$, are eigenfunctions of the laplacian with eigenvalue $-j(j+1)$. Principal continuous representations have

$$j = -\frac{1}{2} + is; \qquad s \in \mathbb{R}\,, \tag{4.1}$$

and are further labelled by a phase $\exp(2\pi i\epsilon)$, where $\epsilon = 0$ in $\mathrm{PSL}(2,\mathbb{R})$, $\epsilon = 0, \frac{1}{2}$ in $\mathrm{SL}(2,\mathbb{R})$, and $\epsilon \in (-\frac{1}{2}, \frac{1}{2}]$ for the universal cover $\widehat{\mathrm{SL}}(2,\mathbb{R})$. The phase $\exp(2\pi i\epsilon)$ corresponds to the representation of the center of the corresponding group. The second class consists of the principal discrete representations, characterized by real $j$, with

$$j \in \mathbb{Z} + \epsilon\,. \tag{4.2}$$

We will choose a basis of eigenvectors of the non-compact $\mathrm{U}(1)$, $g = \exp(\alpha\sigma_3)$. For unitary representations, the corresponding eigenvalue is $\exp(2im\alpha)$, with $m \in \mathbb{R}$. In a given representation, $m$ can take any real value. Moreover, for the continuous representations, there are two vectors with the same value of $m$, which are distinguished by $\pm$.

Waves in the principal continuous series are $\delta$-normalizable; these are the wavefunctions that we shall study in this section. For the continuous representations in the above basis, the non-vanishing matrix elements of $g$ (2.17) are given by:[8]

$$\begin{aligned}\mathbf{K}_{\pm\pm}(\lambda,\mu;j,\epsilon;g) &\equiv \langle j,\epsilon,m,\pm|g|j,\epsilon,\bar{m},\pm\rangle = \\ &= e^{2i(m\alpha+\bar{m}\beta)}e^{2\pi i\epsilon_1\epsilon}\langle j,\epsilon,m,\pm|(i\sigma_2)^{\epsilon_2}g_\delta(\theta)|j,\epsilon,\bar{m},\pm\rangle\,,\end{aligned}\tag{4.3}$$

where

$$\lambda\equiv -im-j\,;\qquad \mu\equiv -i\bar{m}-j\,.\tag{4.4}$$

These matrix elements appear in [17] (for the group $\mathrm{SL}(2,\mathbb{R})$).

Next we shall prepare a wave packet which, after reduction to the coset, describes the scattering of a wave which is incoming from behind the singularity of region 4. In region 4, where $g(y,\theta_4)=e^{\frac{y}{2}\sigma_3}g_4(\theta_4)e^{-\frac{y}{2}\sigma_3}$ with $g_4=-i\sigma_2 g_{1'}=g_1(-i\sigma_2)$ (see eqs. (2.17)–(2.21)and figure 3):

$$g_4=e^{\theta_4\sigma_1}(-i\sigma_2)=\begin{pmatrix}\sinh\theta_4 & -\cosh\theta_4\\ \cosh\theta_4 & -\sinh\theta_4\end{pmatrix}\,,\tag{4.5}$$

the matrix elements in the principle continuous series (with a given $j,\epsilon;m,\bar{m}$) are:

$$\begin{aligned}\mathbf{K}_{+-}(\lambda,\mu;j,\epsilon;g_4) &= \frac{e^{2\pi i\epsilon}}{2\pi i}B(\lambda,-\lambda-2j)\frac{\cosh^{\lambda-\mu}\theta_4}{\sinh^{\lambda-\mu-2j}\theta_4}F\left(\lambda,-\mu-2j;-2j;-\sinh^{-2}\theta_4\right)\,,\\ \mathbf{K}_{-+}(\lambda,\mu;j,\epsilon;g_4) &= \frac{1}{2\pi i}B(1-\mu,\mu+2j+1)\,\times\\ &\quad\times\frac{\cosh^{\lambda-\mu}\theta_4}{\sinh^{\lambda-\mu+2j+2}\theta_4}F\left(1+\lambda+2j,1-\mu;2j+2;-\sinh^{-2}\theta_4\right)\,, \qquad (4.6)\\ \mathbf{K}_{--}(\lambda,\mu;j,\epsilon;g_4) &= \frac{\mathbf{K}_{+-}}{B(\lambda,-\lambda-2j)}\left[B(\lambda,2j+1)+e^{-2\pi i\epsilon}B(-\lambda-2j,1+2j)\right]+\\ &\quad+\frac{\mathbf{K}_{-+}}{B(1-\mu,\mu+2j+1)}\,\times\\ &\quad\times\left[e^{2\pi i\epsilon}B(1+\mu+2j,-2j-1)+B(1-\mu,-2j-1)\right] \qquad (4.7)\\ &\equiv c_{+-}\mathbf{K}_{+-}(\lambda,\mu;j,\epsilon;g_4)+c_{-+}\mathbf{K}_{-+}(\lambda,\mu;j,\epsilon;g_4)\,, \qquad (4.8)\\ \mathbf{K}_{++} &= 0\,, \qquad (4.9)\end{aligned}$$

where $B(a,b)$ is the Euler Beta function

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,,\tag{4.10}$$

and $F(a,b;c;x)$ is the hypergeometric function ${}_2F_1$.

Wave functions (matrix elements) which are incoming only from region 4 have to vanish on the border with region IV (one of the $a=0$ lines in figure 3). For $a=0$ and $\psi\neq 0$ we can choose the gauge

$$g_\gamma=\begin{pmatrix}0 & -1\\ 1 & -\gamma\end{pmatrix}.\tag{4.11}$$

[8]We will sometimes use the label $g$ both for the $2\times 2$ matrices, as well as their representations.

The wave functions $\mathbf{K}_{+-}$, $\mathbf{K}_{-+}$ on that line (between region 4 and IV) are:

$$\begin{aligned}\mathbf{K}_{+-}(\lambda,\mu;j,\epsilon;g_\gamma) &= \frac{e^{2\pi i\epsilon}}{2\pi i}B(\lambda,-\mu-\lambda-2j)\gamma^{\mu+\lambda+2j}\,,\\ \mathbf{K}_{-+}(\lambda,\mu;j,\epsilon;g_\gamma) &= \frac{1}{2\pi i}B(1+\mu+2j,-\lambda-\mu-2j)\gamma^{\mu+\lambda+2j}\,.\end{aligned}\tag{4.12}$$

Therefore, matrix elements in the continuous representations corresponding to wave functions that are incoming from region 4 are given by:

$$\begin{aligned}\mathbf{W}(\lambda,\mu;j,\epsilon;g) =&\frac{2\pi i}{B(\lambda,-\lambda-2j)}\left[e^{-2\pi i\epsilon}\mathbf{K}_{+-}+\frac{\sin(\pi(\mu+2j))}{\sin(\pi\lambda)}\mathbf{K}_{-+}\right]+\\ &+F(\lambda,\mu;j,\epsilon)\mathbf{K}_{++}+G(\lambda,\mu;j,\epsilon)\left[\mathbf{K}_{--}-c_{+-}\mathbf{K}_{+-}-c_{-+}\mathbf{K}_{-+}\right]\,,\end{aligned}\tag{4.13}$$

where $F(\lambda,\mu;j,\epsilon)$ and $G(\lambda,\mu;j,\epsilon)$ are arbitrary (normalizable) functions. A general wave which is incoming from region 4 is given by a wave packet of $\mathbf{W}(\lambda,\mu;j,\epsilon;g)$ in $\lambda,\mu,j$ and $\epsilon$. Since we are interested only in the influence of a scattering from region 4, where $\mathbf{K}_{++}=\mathbf{K}_{--}-c_{+-}\mathbf{K}_{+-}-c_{-+}\mathbf{K}_{-+}=0$ (although not in other regions), we will concentrate on the case where $F=G=0$: When $F$ and/or $G$ are non-zero, the second line in (4.13)describes the physics of wavefunctions entirely outside and independent of the data set in region 4.

The asymptotic behavior of $\mathbf{K}_{+-}$ and $\mathbf{K}_{-+}$ in region 4 is:

$$\begin{aligned}\mathbf{K}_{+-}\left(g_4(\theta_4\to\infty)\right) =&\frac{e^{2\pi i\epsilon}}{2\pi i}B(\lambda,-\lambda-2j)e^{2j\theta_4+2i(m\alpha+\bar m\beta)}\,,\\ \mathbf{K}_{-+}\left(g_4(\theta_4\to\infty)\right) =&\frac{1}{2\pi i}B(1-\mu,\mu+2j+1)e^{-2(j+1)\theta_4+2i(m\alpha+\bar m\beta)}\,,\end{aligned}\tag{4.14}$$

where $\theta_4\to\infty$ is the asymptotically flat boundary of region 4. On the other hand,

$$\begin{aligned}\mathbf{K}_{+-}\left(g_4(\theta_4\to0)\right) &= \frac{e^{2\pi i\epsilon}}{2\pi i}\left(\frac{-b}{c}\right)^{\frac{i}{2}(m-\bar m)}\times\\ &\times\left[B(-\mu-\lambda-2j,\lambda)d^{-i(m+\bar m)}+B(-\lambda-2j,\lambda+\mu+2j)a^{i(m+\bar m)}\right]\,,\\ \mathbf{K}_{-+}\left(g_4(\theta_4\to0)\right) &= \frac{1}{2\pi i}\left(\frac{-b}{c}\right)^{\frac{i}{2}(m-\bar m)}\times\\ &\times\left[B(1+\mu+2j,-\mu-\lambda-2j)d^{-i(m+\bar m)}+B(1-\mu,\lambda+\mu+2j)a^{i(m+\bar m)}\right]\,,\end{aligned}\tag{4.15}$$

where $\theta_4\to0$ is the $a=d=0$ boundary between region 4 and regions I and IV (see eq. (4.5) and figures 3 and 5). Equations (4.13), (4.15) imply:

$$\begin{aligned}\mathbf{W}\left(g_4(\theta_4\to0)\right) &= a^{i(m+\bar m)}\left(\frac{-b}{c}\right)^{\frac{i}{2}(m-\bar m)}\times\\ &\times\left[\frac{B(-\lambda-2j,\lambda+\mu+2j)}{B(\lambda,-\lambda-2j)}+\frac{\sin(\pi(\mu+2j))}{\sin(\pi\lambda)}\frac{B(1-\mu,\lambda+\mu+2j)}{B(\lambda,-\lambda-2j)}\right]\,.\end{aligned}\tag{4.16}$$

It is infinitely blue shifted as $a\to0$: $\mathbf{W}\sim a^{i(m+\bar m)}$, and hence any normalizable wave packet constructed as a superposition of $\mathbf{W}$'s with different values of $m$ and $\bar m$ vanishes at $a=0$, which is the boundary between regions 4 and IV.

On the other hand, in the gauge (2.23)($\alpha = -\beta = y/2$), eqs. (4.13)and (4.14) imply that the behavior of $\mathbf{W}$ in the asymptotically flat region is:

$$\mathbf{W}\left(g_4(\theta_4 \to \infty)\right) \sim e^{-\theta_4 + 2i\omega t}\left[e^{i2s\theta_4} + R(j;m,\bar{m})e^{-i2s\theta_4}\right], \tag{4.17}$$

where

$$j = -\frac{1}{2} + is\,, \qquad t = y\,, \qquad \omega = \frac{1}{2}(m - \bar{m})\,, \tag{4.18}$$

and $R(j;m,\bar{m})$ is given in (3.5). This is precisely the asymptotic behavior of the analytic continuation of the vertex operators in the quotient CFT (see eq. (3.7)). Hence, we conclude that $\mathbf{W}$ corresponds to the analytic continuation of $V$ in the $\frac{\mathrm{SL}(2)\times\mathrm{U}(1)}{\mathrm{U}(1)}$ euclidean CFT.[9] The absolute value of $R$ is

$$|R(j;m,\bar{m})|^2 = \frac{\cosh\left(\pi(2s - m - \bar{m})\right) + \cosh\left(\pi(m - \bar{m})\right)}{\cosh\left(\pi(2s + m + \bar{m})\right) + \cosh\left(\pi(m - \bar{m})\right)}\,. \tag{4.19}$$

For $\omega, s > 0$, the combination $\mathbf{W}$ looks like a plane wave which is incoming from the asymptotically flat boundary of region 4, and scattered from the curved geometry. The damping factor $e^{-\theta_4}$ is cancelled by a corresponding factor in the $\mathrm{SL}(2,\mathbb{R})$ measure. $R(j;m,\bar{m})$ is the reflection coefficient of this scattered wave. Indeed, $|R| \le 1$ as it should in a unitary theory. $R$ is also equal to the two point function of $K_{j;m,\bar{m}}$ (3.3), a primary field of $\mathrm{SL}(2)_L \times \mathrm{SL}(2)_R$ with "spin" $j$ in the $\mathrm{SL}(2)_k$ WZW model in the large $k$ limit (see [19, eq. (3.6)]).

Note that for $s \neq 0$, $R(j;m,\bar{m})$ is a phase if and only if $\bar{m} = -m$. For neutral scattered particles ($(k_L, k_R) = 0$) this condition is satisfied if and only if the black hole is not charged (see eq. (3.11)). We thus learn that an uncharged wavefunction scattered from behind the singularity is fully reflected if and only if the black hole is not charged. The fact that the singularity of the neutral 2-$d$ black hole is a perfect reflector was shown in [3].

The wave function $\mathbf{W}$ can be continued from region 4 to all other regions of the maximally extended 2-$d$ charged black hole. This is done by using the functions $\mathbf{K}$ in the various regions. Some properties of the wave function $\mathbf{W}$ in the extended black hole are presented in the appendix. In particular, it is shown that the total incoming flux from all regions is equal to the total outgoing flux, as expected in a unitary theory (the unitarity being induced by that of SL(2)).

## 5. Summary and discussion

To summarize, the results of section 4 indicate that in the two dimensional charged black hole the regions beyond the singularities should not be ignored, at least in classical string theory. A scattering wave prepared behind the singularity is transmitted ($|R| < 1$), as long as the black hole is charged ($\psi \neq 0 \mod \pi$). Moreover, the scattering wave $\mathbf{W}$ is smooth

[9] The euclidean quotient covers only a single asymptotically flat region of the black hole. Here we have analytically continued to region 4. The continuation to, say, region 1 is done by taking $g \to gi\sigma_2$ in section 3 (thus interchanging, up to signs, the elements $a, b, c, d$ in eq. (3.1)), in which case we shall obtain the analog of the combination $\mathbf{U}$ of [3, 16]. The latter will describe a scattering wave incoming from the asymptotically flat region outside the black hole.

at the singularity, though it is non-analytic (has an infinitely blue shifted piece) at some of the horizons. Infinite blue shifts at the horizons of charged black holes were investigated both for the Reissner-Nordstrom black hole [20, 21] and for two dimensional charged black holes [22, 23]. In both cases, it leads to a large back reaction at the inner horizon and to a potential instability.

We should emphasize that the analysis in this note is based on the assumption that string perturbation theory is valid. Of course, a large back reaction may invalidate this assumption. Indeed, as mentioned above, since any perturbation in the geometry (2.28) is infinitely blue shifted at the inner horizon, a resulting singularity may be expected to form there [20]–[23]. This is one of the reasons why general relativists do not consider seriously the regions behind the inner horizon, where the singularity is located [24]. However, the results in this work show that even if a singularity is formed at the inner horizon, the region beyond it should not necessarily be excluded. As we have seen, there is nothing singular in the already existing singularity, at least as far as low energy scattering from behind it is concerned.

The results in this work are supported by T-duality (for a review, see [5]). An axial-vector abelian duality corresponds to taking $\psi \to \pi - \psi$. In particular, T-duality interchanges the inner horizon at $uv = 1$ with the event horizon at $uv = 0$, and takes the singularity at $uv > 1$ to a line at $uv < 0$, which is a smooth line in the black hole geometry (similar to the behavior of T-duality in the 3-$d$ black string background considered in [8, 12]). Since nothing singular is expected at the $uv < 0$ regions — neither for momentum modes nor for "windings" — such T-duality is compatible with our result that nothing is singular about the singularity. In the uncharged black hole $\psi = 0$ the singularity coincides with the inner horizon at $uv = 1$, and T-duality interchanges the singularity with the event horizon [4, 3]. As we charge the black hole, $\psi$ increases, the singularity is split from the inner horizon and moves towards $uv > 1$ (see eqs. (2.38), (2.39)). In the extremal case $\psi = \frac{\pi}{2}$ (in which case $M = Q$, see eqs. (2.36), (2.34), (2.30)), the singularity in the $u, v$ coordinates approaches $uv \to \infty$, and thus is "removed" (like in [14]),[10] and as $\psi$ turns (formally) bigger then $\pi/2$ the singularity "re-appears" in regions with $uv < 0$. This is all compatible, of course, with T-duality.

Finally, a very closely related two dimensional time-dependent background in the presence of an abelian gauge field is considered in [25]. This 2-$d$ cosmological geometry is described within a family of $\frac{\mathrm{SL}(2)_{k<0}\times \mathrm{U}(1)}{\mathrm{U}(1)\times\mathbb{Z}}$ quotient CFTs. Hence, some of its aspects can be studied by applying the same methods used in [16, 14] and in this note.

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[10] Of course, in the Schwarzschild-like coordinates $r, t$ (2.32)the two horizons coincide, instead.

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## A. Some properties of the scattering wave W

In this appendix we give $\mathbf{W}$ in all the regions of a Poincaré patch: 4',1,1',I and II (region 4 is studied in section 4). In particular, we present the asymptotic behavior of the scattering wave $\mathbf{W}$ in the asymptotically flat regions, and check that the total incoming flux is equal to the total outgoing flux.

Between asymptotically flat regions $a$ and $a'$, $a = 1, \ldots, 4$, in $\mathrm{SL}(2,\mathbb{R})$ we have the relation $g_{1'}(\theta) = g_1(-\theta) = -i\sigma_2 g_1(\theta) i\sigma_2$ (see (2.20)), and between the "intermediate" regions $i = I, \ldots, IV$ the relation is $g_I(-\theta) = \sigma_3 g_I(\theta)\sigma_3$ (see (2.19)). The relations in regions with $a > 1$ and $i > I$ follow from eqs. (2.17)–(2.20).[11] In particular, these relations give:

$$\mathbf{K}_{\pm\mp}(\lambda, \mu; j, \epsilon; g_a) = \mathbf{K}_{\mp\pm}(\lambda, \mu; j, \epsilon; g_{a'})\,, \qquad a = 1, 2, 3, 4\,. \tag{A.1}$$

We will use (A.1) to read the behavior of $\mathbf{W}$ in region $a'$ from its behavior in region $a$.

**In region 4' we have:**

$$\mathbf{W}\left(g_{4'}(\theta_{4'} \to \infty)\right) = e^{-\theta_{4'} - 2i\omega t}\left[A_{\text{in } 4'} e^{i2s\theta_{4'}} + A_{\text{out } 4'} e^{-i2s\theta_{4'}}\right], \tag{A.2}$$

where[12]

$$A_{\text{in } 4'} = e^{2\pi i\epsilon}\frac{\sin(\pi(\mu + 2j))}{\sin(\pi\lambda)}\,, \quad A_{\text{out } 4'} = -e^{-2\pi i\epsilon}\frac{\Gamma(1-\mu)\Gamma(1+\mu+2j)\Gamma(-2j-1)}{\Gamma(\lambda)\Gamma(-\lambda-2j)\Gamma(2j+1)} \tag{A.3}$$

$$|A_{\text{in } 4'}|^2 = \frac{\cosh(\pi(\bar{m} - s))^2}{\cosh(\pi(m + s))^2}\,, \quad |A_{\text{out } 4'}|^2 = \frac{\cosh(\pi(m - s))\cosh(\pi(m + s))}{\cosh(\pi(\bar{m} - s))\cosh(\pi(\bar{m} + s))}\,. \tag{A.4}$$

**In region 1 we have:**

$$\begin{aligned}\mathbf{K}_{-+}(\lambda, \mu; j, \epsilon; e^{\theta_1\sigma_1}) = {} & \frac{1}{2\pi i}\cosh^{2j+\lambda+\mu}\theta_1 \times \\ & \times \left[B(\lambda, 1-\mu)\sinh^{\lambda-\mu}\theta_1 F\left(\lambda, \lambda + 2j + 1; \lambda - \mu + 1; -\sinh^2\theta_1\right) + \right. \\ & \quad + e^{2\pi i\epsilon} B(-\lambda - 2j, \mu + 2j + 1)\sinh^{\mu-\lambda}\theta_1 \times \\ & \quad \left.\times\, F\left(\mu, \mu + 2j + 1; \mu - \lambda + 1; -\sinh^2\theta_1\right)\right]\,,\end{aligned} \tag{A.5}$$

$$\mathbf{K}_{+-}(\lambda, \mu; j, \epsilon; e^{\theta_1\sigma_1}) = 0\,. \tag{A.6}$$

The asymptotic behavior in region 1 is

$$\begin{aligned}\mathbf{W}\left(g_1(\theta_1 \to \infty)\right) &= \frac{2\pi i\sin(\pi(\mu + 2j))}{\sin(\pi\lambda)B(\lambda, -\lambda - 2j)}\mathbf{K}_{-+}\left(g_1(\theta_1 \to \infty)\right) \\ &= e^{-\theta_1 + 2i\omega t}\left[e^{i2s\theta_1} A_{\text{in } 1}(j; m, \bar{m}) + e^{-i2s\theta_1} A_{\text{out } 1}(j; m, \bar{m})\right],\end{aligned} \tag{A.7}$$

[11] In $\widehat{\mathrm{SL}}(2,R)$ these relations hold up to a jump between different copies (which is the lift of $e^{2\pi i\sigma_2}$) and give an additional factor of $e^{4\pi i\epsilon}$ to the matrix element.

[12] The interpretation of incoming is for incoming positive energy particles (creation operators) or outgoing negative energy particles (annihilation operators).

where

$$A_{in\ 1}(j;m,\bar{m}) = \frac{\sin(\pi(\mu+2j))}{\sin(\pi\lambda)} \frac{\Gamma(2j+1)}{B(\lambda,-\lambda-2j)} \left[\frac{\Gamma(\lambda)}{\Gamma(\lambda+2j+1)} + e^{2\pi i\epsilon}\frac{\Gamma(-\lambda-2j)}{\Gamma(1-\lambda)}\right]$$

$$A_{\rm out\ 1}(j;m,\bar{m}) = \frac{\sin(\pi(\mu+2j))}{\sin(\pi\lambda)} \frac{\Gamma(-2j-1)}{B(\lambda,-\lambda-2j)} \left[\frac{\Gamma(1-\mu)}{\Gamma(-\mu-2j)} + e^{2\pi i\epsilon}\frac{\Gamma(\mu+2j+1)}{\Gamma(\mu)}\right] \quad (A.8)$$

$$|A_{\rm in\ 1}|^2 = \frac{\cosh(\pi(\bar{m}-s))^2}{\cosh(\pi(m+s))^2} \frac{|\cosh(\pi(m+s)) + e^{2\pi i\epsilon}\cosh(\pi(m-s))|^2}{\sinh^2(2\pi s)}$$

$$|A_{\rm out\ 1}|^2 = \frac{\cosh(\pi(s-m))\cosh(\pi(s-\bar{m}))}{\cosh(\pi(s+m))\cosh(\pi(s+\bar{m}))} \frac{|\cosh(\pi(\bar{m}+s)) + e^{2\pi i\epsilon}\cosh(\pi(\bar{m}-s))|^2}{\sinh^2(2\pi s)}. \quad (A.9)$$

**In region 1' we have:**

$$A_{in\ 1'}(j;m,\bar{m}) = e^{-2\pi i\epsilon} \frac{\Gamma(2j+1)}{B(\lambda,-\lambda-2j)} \left[\frac{\Gamma(\lambda)}{\Gamma(\lambda+2j+1)} + e^{2\pi i\epsilon}\frac{\Gamma(-\lambda-2j)}{\Gamma(1-\lambda)}\right]$$

$$A_{\rm out\ 1'}(j;m,\bar{m}) = e^{-2\pi i\epsilon} \frac{\Gamma(-2j-1)}{B(\lambda,-\lambda-2j)} \left[\frac{\Gamma(1-\mu)}{\Gamma(-\mu-2j)} + e^{2\pi i\epsilon}\frac{\Gamma(\mu+2j+1)}{\Gamma(\mu)}\right] \quad (A.10)$$

$$|A_{\rm in\ 1'}|^2 = \frac{|\cosh(\pi(m+s)) + e^{2\pi i\epsilon}\cosh(\pi(m-s))|^2}{\sinh^2(2\pi s)} \quad (A.11)$$

$$|A_{\rm out\ 1'}|^2 = \frac{\cosh(\pi(s+m))\cosh(\pi(s-m))}{\cosh(\pi(s+\bar{m}))\cosh(\pi(s-\bar{m}))} \frac{|\cosh(\pi(\bar{m}+s)) + e^{2\pi i\epsilon}\cosh(\pi(\bar{m}-s))|^2}{\sinh^2(2\pi s)}.$$

**In region I we have ($-\frac{\pi}{2} < \theta_I < 0$):**

$$\begin{aligned} &\mathbf{W}(\lambda,\mu;j,\epsilon;e^{i\theta_I\sigma_2}) = \quad (A.12)\\ &= \frac{B(-\lambda-2j,\mu+2j+1)}{B(\lambda,-\lambda-2j)} \frac{\sin^{\mu-\lambda}\theta_I}{\cos^{\mu-\lambda-2j}\theta_I} F\left(-\lambda-2j,\mu;\mu-\lambda+1;-\tan^2\theta_I\right) + \\ &\quad + \frac{\sin(\pi(\mu+2j))}{\sin(\pi\lambda)} \frac{B(\lambda,1-\mu)}{B(\lambda,-\lambda-2j)} \frac{\sin^{\lambda-\mu}\theta_I}{\cos^{\lambda-\mu-2j}\theta_I} F\left(\lambda,-\mu-2j;\lambda-\mu+1;-\tan^2\theta_I\right). \end{aligned}$$

**In region II we have ($0 < \theta_{II} < \frac{\pi}{2}$):**

$$\begin{aligned} \mathbf{W}(\lambda,\mu;j,\epsilon;e^{i\theta_{II}\sigma_2}) &= e^{-2\pi i\epsilon} \frac{B(\lambda,1-\mu)}{B(\lambda,-\lambda-2j)} \frac{\sin^{\lambda-\mu}(-\theta_{II})}{\cos^{\lambda-\mu-2j}\theta_{II}} \times \\ &\quad \times F\left(\lambda,-\mu-2j;\lambda-\mu+1;-\tan^2\theta_{II}\right) + \\ &\quad + e^{2\pi i\epsilon} \frac{\sin(\pi(\mu+2j))}{\sin(\pi\lambda)} \frac{B(-\lambda-2j,\mu+2j+1)}{B(\lambda,-\lambda-2j)} \times \\ &\quad \times \frac{\sin^{\mu-\lambda}(-\theta_{II})}{\cos^{\mu-\lambda-2j}\theta_{II}} F\left(-\lambda-2j,\mu;\mu-\lambda+1;-\tan^2\theta_{II}\right). \quad (A.13) \end{aligned}$$

The behavior of $\mathbf{W}$ in all other regions in $\widehat{\rm SL}(2,\mathbb{R})$ is obtained by the relation[13]

$$\mathbf{K}_{\pm\pm}(\lambda,\mu;j,\epsilon;-g) = e^{2\pi i\epsilon}\mathbf{K}_{\pm\pm}(\lambda,\mu;j,\epsilon;g). \quad (A.14)$$

[13] And the phase $e^{4\pi i\epsilon}$ that the wavefunction acquires when we go from one copy of $\mathrm{SL}(2,\mathbb{R})$ to the next one in the universal cover.

Note that, for every $m, \bar{m}, s$ and $\epsilon$, the total incoming flux is equal to the total outgoing flux, namely:

$$\left(1+|A_{\text{in }4'}|^2+|A_{\text{in }1}|^2+|A_{\text{in }1'}|^2\right)-\left(|R|^2+|A_{\text{out }4'}|^2+|A_{\text{out }1}|^2+|A_{\text{out }1'}|^2\right)=0\,, \tag{A.15}$$

where we have normalized $A_{\text{in }4}=1$, hence $R\equiv A_{\text{out }4}$. Finally, for $\bar{m}=-m$ ($\psi=0$) we have $|A_{\text{in }4}|^2=|A_{\text{in }4'}|^2=|A_{\text{out }4'}|^2=|R|^2=1$, and for $\epsilon=0$, $\bar{m}=m$ ($\psi=\pi$) $\mathbf{W}$ vanish in region II.

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